On Generalized Fuzzy n –Normed Spaces Including φ Function

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Abstract— The fundamental aim of this paper is to consider and introduce fuzzy φ-n-normed space where φ function is introduced originally by Golet in [6].

Index Terms— 2-normed spaces, n-normed spaces, fuzzy normed spaces.

1 Introduction

The concept of 2-normed space was firstly introduced by Gahler in [1]. Bag and Samanta also introduced and developed the fuzzy normed spaces in [3], [4] which are important to study in fuzzy systems. In [5], Narayanan and Vijayabalaji gave fuzzy n-normed space.

In the value of dimensions $n=1,2$ Golet considered a generalization of fuzzy normed space in [6] using a real valued function φ as well.

In the present paper we will introduce the concept of fuzzy φ-n-normed spaces as a generalization of fuzzy n-normed space.

Definition 1. [6] Let φ be a function defined on the real field $\mathbb{R}$ into itself with the following properties:

1) $\varphi(-t)=\varphi(t)$, for all $t\in\mathbb{R}$,
2) $\varphi(1)=1$,
3) φ is strict increasing and continuous on $(0,\infty)$,
4) $\lim_{t\to 0}\varphi(t)=0$ and $\lim_{t\to\infty}\varphi(t)=0$.

Now, we will give an example on the above definition:

1) $\varphi(t)=|t|$,
2) $\varphi(t)=|t|^p\ p\in\mathbb{R}_+$.

Definition 2. [6] A $t-norm$ $*$ is a two place function $*:[0,1]\times[0,1]\to[0,1]$ which is associative, commutative, nondecreasing in each place and that $a*1=a$ for all $a\in[0,1]$.

The most used $t-norms$ in fuzzy metric spaces are following:

1) $a*b=ab$,
2) $a*b=\min\{a,b\}$,
3) $a*b=\max\{a+b-1,0\}$.

Definition 3. [6] By an operation " o" on $\mathbb{R}_+$ we mean a two place function $o:[0,\infty)\times[0,\infty)\to[0,\infty)$ which is associative, commutative, nondecreasing in each place and such that $ao0=a$ for all $a\in[0,\infty)$. The most used operations on $\mathbb{R}_+$ are following:

1) $o(s,t)=t+s$,
2) $o(s,t)=\max\{s,t\}$,
3) $o(s,t)=(s^n+t^n)^{1/n}$

2 Main Result

In this section , we will give the definition of generalized fuzzy $\varphi-n-normed$ *space.*

Definition 4. Let $n\in\mathbb{N}$ and X be a real vector space of dimension $d\geq n$. A real valued function $\|.,\dots,\|$ on X^n satisfying the following

1) $\|x_1,\dots,x_n\|=0$ iff $x_1,\dots,x_n$ are linearly dependent,
2) $\|x_1,\dots,x_n\|$ is invariant under permutation,
3) $\|x_1,\dots,cx_n\|=\varphi(c)\|x_1,\dots,x_n\|$ for all $c\in\mathbb{R}$,
4) $\|x_1,\dots,x_{n-1},y+z\|\leq\|x_1,\dots,x_{n-1},y\|+\|x_1,\dots,x_{n-1},z\|$

is called an $\varphi-n-norm$ on X and the pair $(X,\|.,\dots,\|)$ is called $\varphi-n-normed$ *space.*

Corollary 1. When we take $\varphi(t)=|t|$ then Definition 4 reduces the definition of Gunawan and Mashadi [2].

Corollary 2. When we take only $n=1,2$ then we obtain the definition given by Golet [6],[7].

Corollary 3. Also, if we take $\varphi(t)=|t|$ and $n=2$ then we obtain the definition given by Gahler [1].

Definition 5. Let X be a linear space over real field $\mathbb{R}$ of dimension $d\geq n$ and let N be a mapping defined on $X^n\times[0,\infty)$ with values into $[0,1]$ satisfying the following conditions: for all $x_1,\dots,x_n,x,y\in X$ and all $s,t\in[0,\infty)$

F1) $N(x_1,\dots,x_n,0)=0$,
F2) for all $t>0$ $N(x_1,\dots,x_n,t)=1$ if and only if $x_1,\dots,x_n$ are linearly dependent,
F3) $N(x_1,\dots,x_n,t)$ is invariant under any permutation of $x_1,\dots,x_n$,
F4) $N(x_1,\dots,x_n,ct)=N\left(x_1,\dots,x_n,\frac{t}{\varphi(t)}\right),\ \varphi(c)\neq 0$,
F5) $N(x_1,\dots,x_n,.)$ is nondecreasing function on $[0,\infty)$ and $\lim_{n\to\infty}N(x_1,\dots,x_n,t)=1$
F6) $N(x,y,\dots,x_n,tos)\geq N(x,\dots,x_n,t)*N(y,\dots,x_n,s)$.

The triple $(X,N,*)$ is called *generalized fuzzy* $\varphi-n-normed$ *space.*

Corollary 4. Substituting $o(s,t)=t+s$, $a*b=\min\{a,b\}$ and $\varphi(t)=|t|$ in Definition 5 then , the triple $(X,N,*)$ is called fuzzy $n-normed$ *space* which is defined by Narayanan and Vijayabaliji in [5].

Corollary 5. When we only consider $n=1,2$ in Definition 5, we obtain definition of fuzzy $\varphi-normed$ and fuzzy $\varphi-2-normed$ *space* that given by Golet [6,7].

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Corollary 5. Also, for $n = 1$ if we take $o(s,t) = t + s$, $a * b = \min\{a,b\}$, and $\varphi(t) = |t|$ then we obtain definition of fuzzy normed space which is given by T. Bag and S.K. Samanta[1,3].

Example 1. Let $(X, \|.,\ldots,\|)$ be $\varphi - n - normed$ space. For all $x_1, \ldots, x_n \in X$,$t \in [0,\infty)$

$$N(x_1, \ldots, x_n, t) = \frac{t}{t + \|x_1, \ldots, x_n\|}$$

then, $(X, N,*)$ is generalized fuzzy $\varphi - n - normed$ space. In the case of $o(s,t) = t + s$, $a * b = \min\{a,b\}$ then we calls $(X, N,*)$ as the standard fuzzy $\varphi - n - normed$ space. Also, if we take $\varphi(t) = |t|$ then $(X, N,*)$ is called standard fuzzy $n - normed$ space.

Proof. It is clear that F1, F2, F3, F5 and F6 are satisfied for all $x_1, \ldots, x_n \in X$. So, it is necessary to show F4 as the following:
For $c \in \mathbb{R}$ and $\varphi(c) \neq 0$, we easily see that

$$N(x_1, \ldots, x_n, ct) = \frac{ct}{ct + \|x_1, \ldots, x_n\|} = N\left(x_1, \ldots, x_n, \frac{t}{\varphi(t)}\right)$$

Thus, the proof is completed.

Corollary 6. Let $(X, N,*)$ be an generalized fuzzy $\varphi - n - normed$ space. By the property $\varphi(-t) = \varphi(t)$, for all $x_1, \ldots, x_n \in X$,$t \in [0,\infty)$ we obtain that

$$N(y - x, x_2, \ldots, x_n, t) = N(x - y, x_2, \ldots, x_n, t).$$

Corollary 7. . Let $(X, N,*)$ be an generalized fuzzy $\varphi - n - normed$ space by F3) and F4) for all $x_1, \ldots, x_n \in X$,$t \in [0,\infty)$ we obtain that

$$N(cx_1, \ldots, cx_n, t) = N\left(x_1, \ldots, x_n, \frac{t}{\varphi(t)^n}\right)$$

Theorem 1. Let $(X, N,*)$ be an generalized fuzzy $\varphi - n - normed$ space with assumption
F7) $N(x_1, \ldots, x_n, t) > 0$ for all $t > 0$ implies $x_1, \ldots, x_n$ are linearly dependent. Define :

$$\|x_1, \ldots, x_n\|_{\varphi(\alpha)} = \inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha), \alpha \in (0,1)\}$$

then, $\{\|.,\ldots,.\|_{\varphi(\alpha)}: \alpha \in (0,1)\}$ is an ascending family of $\varphi - n - normed$ space on X. They are called $\varphi(\alpha) - norm$ on X corresponding to the fuzzy $\varphi - n - norm$ on X.

Proof. Firstly we emphasize that since $\alpha \in (0,1)$ and from definition φ we have $\varphi(\alpha) \in (0,1)$. Now, we prove our theorem systematically as follows:

1) It is easy to see that
$$\|x_1, \ldots, x_n\|_{\varphi(\alpha)} = 0$$
This implies,
$$\inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha), \alpha \in (0,1)\} = 0.$$
This implies
$$\text{For all } t > 0,\ N(x_1, \ldots, x_n, t) \geq \varphi(\alpha) > 0$$
This implies, by F7) $x_1, \ldots, x_n$ are linearly dependent.
Conversely assume that $x_1, \ldots, x_n$ are linearly dependent.
This implies, by F2),
$$N(x_1, \ldots, x_n, t) = 1 \text{ for all } t > 0.$$
This implies,
$$\inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha), \alpha \in (0,1)\} = 0.$$
This implies,
$$\|x_1, \ldots, x_n\|_{\varphi(\alpha)} = 0.$$

2) Since $N(x_1, \ldots, x_n, t)$ is invariant under permutation so $\|x_1, \ldots, x_n\|_{\varphi(\alpha)}$ is invariant under permutation.

3) For all $c \in \mathbb{R}$, $\alpha \in (0,1)$ and $s \in [0,\infty)$, we acquire

$$\|x_1, \ldots, cx_n\|_{\varphi(\alpha)} = \inf\{s: N(x_1, \ldots, cx_n, s) \geq \varphi(\alpha)$$

$$= \inf\{s: N\left(x_1, \ldots, x_n, \frac{s}{\varphi(c)}\right) \geq \varphi(\alpha).$$

By taking $t = \frac{s}{\varphi(c)} \in [0,\infty)$, then, we have

$$\|x_1, \ldots, cx_n\|_{\varphi(\alpha)} = \inf\{t\varphi(c): N(x_1, \ldots, x_n, t) \geq \varphi(\alpha)\}$$

$$= \varphi(c).\inf\{t: N(x_1, \ldots, x_n, s) \geq \varphi(\alpha)\}$$
$$= \varphi(c)\|x_1, \ldots, x_n\|_{\varphi(\alpha)}.$$

4) $\|x_1, \ldots, x_n\|_{\varphi(\alpha)} + \|x_1, \ldots, x_n'\|_{\varphi(\alpha)}$
$$= \inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha)\}$$
$$+ \inf\{s: N(x_1, \ldots, x_n', s) \geq \varphi(\alpha)\}$$
$$\geq \inf\{t + s: N(x_1, \ldots, x_n + x_n', t + s) \geq \varphi(\alpha)\}$$
$$= \inf\{k: N(x_1, \ldots, x_n + x_n', k) \geq \varphi(\alpha), k = t + s\}$$
$$= \|x_1, \ldots, x_n + x_n'\|_{\varphi(\alpha)}.$$

Therefore
$$\|(x_1, \ldots, x_n + x_n', k)\|_{\varphi(\alpha)} \leq \|x_1, \ldots, x_n\|_{\varphi(\alpha)} + \|x_1, \ldots, x_n'\|_{\varphi(\alpha)}$$
So, $\|.,\ldots,.\|_{\varphi(\alpha)}$ is $\varphi - n - norm$ on X.
Now, we show that for any $\alpha_1 < \alpha_2 \in (0,1)$

$$\|x_1, \ldots, x_n\|_{\varphi(\alpha_1)} < \|x_1, \ldots, x_n\|_{\varphi(\alpha_2)}.$$

Since $\alpha_1 < \alpha_2$ by definition of φ, we obtain that
$$\varphi(\alpha_1) < \varphi(\alpha_2)$$

$$\|x_1, \ldots, x_n\|_{\varphi(\alpha_1)} = \inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha_1)\}$$

$$\|x_1, \ldots, x_n\|_{\varphi(\alpha_2)} = \inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha_2)\}$$

Thus, we deduce:

$$\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha_2)\} \subset \{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha_1)\}$$

This implies that

$$\inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha_2)\} \geq inf\{t: N(x_1, \ldots, x_n, t) \geq \varphi(\alpha_1)\}$$

Thus, we obtain that

$$\|x_1, \ldots, x_n\|_{\varphi(\alpha_1)} < \|x_1, \ldots, x_n\|_{\varphi(\alpha_2)}.$$

Consequently, the proof is completed.

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